Research Statement

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A mantra in modern algebraic geometry is "objects are best studied in families." Moduli spaces embrace this in the fullest sense, revealing how varieties or schemes behave and relate to each other as they deform or degenerate. A prime example is $\overline{\mathcal{M}}_{g,n}$, which parametrizes stable n-marked curves of genus g, and which, as a compactification of the space of smooth curves $\mathcal{M}_{g,n}$, gives insight into degenerations of smooth curves in families. Since curves naturally appear in many different contexts, $\overline{\mathcal{M}}_{g,n}$ serves as a testing ground for exploration in diverse areas.

One natural goal when studying a space like $\overline{\mathcal{M}}_{g,n}$ is to consider other spaces which are birational to it, that is, spaces which are very similar but different in subtle ways. Investigating the differences can then give new information and interpretations of the original space. To this end, studying alternative compactifications of $\mathcal{M}_{g,n}$ birational to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ has proven to be a fruitful endeavor. My research explores and utilizes such compactifications, their inherent combinatorial structure, and how they related to each other in order to gain deeper insight into families of curves.

My work can broadly be grouped according to the following questions.

- How does the intersection theory of $\overline{\mathcal{M}}_{g,n}$ relate to other birational compactifications?
- Can we classify all possible compactifications of $\mathcal{M}_{g,n}$?
- What do the cones of effective classes on the moduli space of curves look like?

Families of curves are a natural starting place to test and frame these questions, but they are readily generalized to higher-dimensional varieties as well.

One recurring theme of my research is the utilization of combinatorics to make difficult geometric problems more tractable. What makes this possible is the recursive structure of the moduli space of curves: certain subspaces of $\overline{\mathcal{M}}_{g,n}$ called *boundary strata* are isomorphic to products of smaller moduli spaces of curves $\prod \overline{\mathcal{M}}_{g_i,n_i}$. These subspaces stratify $\overline{\mathcal{M}}_{g,n}$ by topological type, which allows us to associate to each boundary stratum (and to each curve) a *dual graph* (see Figure 1).

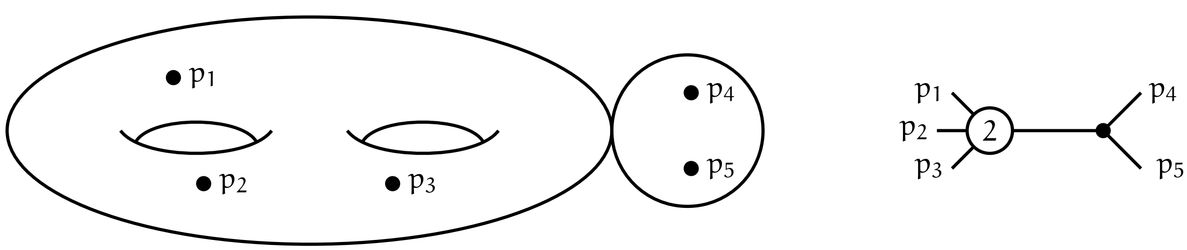

Figure 1. A nodal 5-marked curve and its dual graph representation.

Dual graphs are incredibly useful; for example, the intersection theory of $\overline{\mathcal{M}}_{g,n}$ can be framed in terms of manipulations of graphs, and entire families of compactifications can be defined just in terms of specified collections of graphs. This graphical interpretation of the boundary of $\overline{\mathcal{M}}_{g,n}$ has intriguing connections to surprising areas of math such as information theory and evolutionary dynamics via an association with phylogenetic trees.

1. Intersection Theory and Alternate Compactifications

In [BC18], Cavalieri and I begin the investigation of the intersection theory of particular tautological classes, called *ω-classes*. Let $\rho_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,1}$ be the rememberful morphism which forgets all markings except the ith. We prove the following explicit graph formula for arbitrary monomials of ω-classes on $\overline{\mathcal{M}}_{g,n}$ (the $\Delta_{\mathcal{P}}$ refer to specific boundary strata).

Theorem 6 ([BC18]). *For $1 \leq i \leq n$, let k_i be a non-negative integer, and let $K = \sum_{i=1}^{n} k_i$. For any partition $\mathcal{P} = \{P_1, \ldots, P_r\} \vdash [n]$, define $\alpha_j := \sum_{i \in P_j} k_i$. With $\Delta_{\mathcal{P}}$ the pinwheel stratum defined by $\mathcal{P}$, the following formula holds in $R^K(\overline{\mathcal{M}}_{g,n})$:*

$$\prod_{i=1}^{n} \omega_i^{k_i} = \sum_{\mathcal{P} \vdash [n]} [\Delta_{\mathcal{P}}] \prod_{j=1}^{\ell(\mathcal{P})} \frac{\psi_{\bullet_j}^{\alpha_j}}{(-\psi_{\bullet_j} - \psi_{\star_j})^{1-\delta_j}},$$

where $\delta_j = \delta_{1,|P_j|}$ is a Kronecker delta and we follow the standard convention of considering negative powers of ψ equal to 0.

We also deduce a numerical corollary when the intersection is top-dimensional.

Corollary 7 ([BC18]). *For $1 \leq i \leq n$, let k_i be a non-negative integer, and let $\sum_{i=1}^{n} k_i = 3g - 3 + n$. For any partition $\mathcal{P} = \{P_1, \ldots, P_r\} \vdash [n]$, define $\alpha_j := \sum_{i \in P_j} k_i$. Then*

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \omega_i^{k_i} = \sum_{\mathcal{P} \vdash [n]} (-1)^{n+\ell(\mathcal{P})} \int_{\overline{\mathcal{M}}_{g,\ell(\mathcal{P})}} \prod_{i=1}^{\ell(\mathcal{P})} \psi_{\bullet_i}^{\alpha_i - |P_i| + 1}.$$

These ω-classes can be seen as coming from a special type of compactification of $\mathcal{M}_{g,n}$, called a *Hassett space*, or moduli spaces of weighted pointed stable curves, first defined in [Has03]. This family of compactifications, introduced in [Has03], comes by way of *Hassett spaces*, or moduli spaces of weighted pointed stable curves. Define *weight data* $\mathcal{A} = (a_1, \ldots, a_n)$ as a collection of rational numbers in the interval $(0, 1]$ such that $2g - 2 + \sum a_i > 0$. Then a genus g pointed curve $(C; p_1, \ldots, p_n)$ is *$\mathcal{A}$-stable* if

- C has at worst nodal singularities, with $p_i \in C$ smooth;
- the divisor $K_C + a_1 p_1 + \cdots + a_n p_n$ is ample;
- if a collection of the p_i coincide, the sum of the corresponding a_i is less than or equal to one.

In the Deligne-Mumford compactification, when two points wish to collide, they instead "bubble-off" to a new rational component; in Hassett spaces, two points are instead allowed to collide so long as their weights are not too high. Given the poset structure on weight datas of the same length determined by $\mathcal{A} \leq \mathcal{B}$ if $a_i \leq b_i$ for all i, we have reduction morphisms $r : \overline{\mathcal{M}}_{g,\mathcal{B}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ which lower the weights of $\mathcal{B}$ to those of $\mathcal{A}$.

An important observation is that tautological classes on $\overline{\mathcal{M}}_{g,\mathcal{A}}$ pull back under r to tautological classes on $\overline{\mathcal{M}}_{g,\mathcal{B}}$ in highly combinatorial ways. In particular, for the cotangent classes ψ_i,

$$r^*\psi_i = \psi_i - D_{\mathcal{B},\mathcal{A}},$$

where $D_{\mathcal{B},\mathcal{A}}$ parametrizes all $\mathcal{B}$-stable curves such that the ith marked point lives on a rational component which becomes unstable with the $\mathcal{A}$ weights.

Our ω-classes are identified as pullbacks from a *heavy-light Hassett space* of light-weight ψ-classes. In the heavy-light case, the weight data is given by $(1^{(n)}, \epsilon^{(m)})$. In these spaces, any number of ϵ-weighted marked points (light points) may come together, but none may collide with the 1-weighted points (heavy points). In [BC20], we generalize Theorem 6 to the case of arbitrary weight data, allowing us to write any arbitrary intersection of weighted ψ-classes in terms of a combinatorial expression of unweighted ψ-classes.

One way to package these intersection numbers is through generating functions. The genus g Gromov-Witten potential of a point is defined to be

$$\mathcal{F}_g(t_0, t_1, \ldots) = \langle e^{\vec{\tau}} \rangle_g = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \vec{\tau}, \ldots, \vec{\tau} \rangle_{g,n}$$

where $\langle \vec{\tau}, \ldots, \vec{\tau} \rangle_{g,n}$ is the Witten bracket which corresponds to a particular integral of ψ-classes. The total Gromov-Witten potential $\mathcal{F}$ is obtained by summing over all genera and adding a formal variable to keep track of genus.

In the case of no marked points, we may define the κ-classes as pushforwards of powers of ψ-classes under forgetful morphisms. We may define a κ-potential $\mathcal{K}$ similarly to the Gromov-Witten potential above, exchanging s_i for t_i and σ_i for τ_i, so that the coefficients of $\mathcal{K}$ will correspond to monomials of the form $\kappa_0^{\ell_0} \cdots \kappa_{3g-3}^{\ell_{3g-3}}$. In [BC21], we prove the following.

Theorem 2 ([BC21])**.** *The κ-potential $\mathcal{K}$ is a change of variables of $\mathcal{F}$ given by $s_i = S_i(t_{i+1})$, where S_i is the ith elementary Schur polynomial.*

This theorem recovers a result in [MZ00], though our methods differ considerably. Further, we are able to apply the change of variables Witten's conjecture (Kontsevich's theorem by [Kon92]) that the Witten potential is annihilated by the Virasoro operators, allowing us to compute the κ-class Virasoro operators.

Corollary 3 ([BC21])**.** *The κ-potential is annihilated by appropriately transformed Virasoro operators.*

1.1. **Work in progress.** A partial classification of all sufficiently nice compactifications of $\mathcal{M}_{g,n}$ is given in combinatorial terms in [Smy13], and in [BKN21] the authors use related ideas to completely classify all modular compactifications by Gorenstein curves with distinct marked points in genus one via Q-stable spaces $\overline{\mathcal{M}}_{1,n}(Q)$, dependent on partition data Q. In ongoing work with Bozlee, we achieve an analogous result in genus zero while allowing marked points to collide, using a generalization of Hassett spaces that depend on a simplicial complex K on the set $\{1, \ldots, n\}$. In this setup, a collection of marked points $S \subseteq \{p_1, \ldots, p_n\}$ are allowed to collide if and only if all indices in S share a simplex in K. We show that in genus zero, the spaces $\overline{\mathcal{M}}_{0,K}$ recover all modular (i.e., points in the boundary have a natural modular interpretation) compactifications by Gorenstein curves with smooth markings.

Theorem 5 ([BB])**.** *If $\mathcal{M}$ is a modular compactification of $\mathcal{M}_{0,n}$ by Gorenstein curves with smooth markings, then $\mathcal{M} \cong \overline{\mathcal{M}}_{0,K}$ for some simplicial complex K.*

We also combine the compactifications in [BKN21] with the K-compactifications, and we conjecture that these combined spaces complete the classification in genus one.

Conjecture 8 ([BB]). *If $\mathcal{M}$ is a modular compactification of $\mathcal{M}_{1,n}$ by Gorenstein curves, then $\mathcal{M} \cong \overline{\mathcal{M}}_{1,K}(Q)$ for some simplicial complex K and some partition data Q.*

In a related ongoing project, Bozlee and I show that the ψ-class intersection theory of $\overline{\mathcal{M}}_{g,K}$ in arbitrary genus is a natural generalization of that of Hassett spaces, and we have a conjectural description of the parallel theory for $\overline{\mathcal{M}}_{g,K}(Q)$. This generalizes [Smy19], in which the ψ-class theory is worked out for a special subfamily of Q-stable spaces.

2. Extremal Classes

The cone of effective divisors on a projective variety X dictates its birational geometry, and when X is a moduli space, birational models of X often have new modular interpretations and useful connections to each other. For this reason and others, the structure of the cone of effective divisors of $\overline{\mathcal{M}}_{g,n}$ has attracted a great deal of attention, in for example [CC14, CT15, Opi16, Mul17b, Mul20]. More generally, there has been interest in probing the finer aspects of the birational geometry of moduli spaces by studying the cones of effective higher codimension cycles, e.g., [Mul17a, Che18, Mul19]. Cones are naturally defined by their extremal rays, and determining which classes are extremal in these cones is one avenue of investigation. An effective class α on a space X is called *extremal* if for every effective decomposition

$$\alpha = \sum a_i [E_i],$$

the classes $[E_i]$ are all proportional to α. My research has studied two large families of effective classes on $\overline{\mathcal{M}}_{g,n}$ in this context: hyperelliptic classes and boundary strata.

The *hyperelliptic locus* in $\overline{\mathcal{M}}_{g,n}$, the subset of the moduli space corresponding to curves which admit a degree two map to $\mathbb{P}^1$, has been defined and analyzed in several ways. In [EH87, Tar15, CT16, CC15], the hyperelliptic loci have marked points at the Weierstrass points; less commonly, the marks are placed on conjugate pairs of points [Log03]. In [CT16], Chen and Tarasca show that genus-two hyperelliptic classes with marked Weierstrass points are rigid and extremal in the cone of effective classes on $\overline{\mathcal{M}}_{2,n}$. In [Bla20], I generalize this result by allowing hyperelliptic loci to have marked Weierstrass points, marked conjugate pairs, and condition-free marked points. This increase in flexibility allows me to prove the following theorem.

Theorem 9 ([Bla20]). *The class $[\overline{\mathcal{H}}_{2,\ell,2m,n}]$ of hyperelliptic curves in genus two with ℓ marked Weierstrass points, m marked conjugate pairs, and n condition-free marked points is rigid and extremal in the pseudo-effective cone of codimension-$(\ell+m)$ classes on $\overline{\mathcal{M}}_{2,\ell+2m+n}$ for all $\ell, m, n \geq 0$.*

Some of the techniques utilized in proving Theorem 9 lend themselves to other families of effective classes on $\overline{\mathcal{M}}_{g,n}$, including the boundary strata defined above. All boundary divisors on $\overline{\mathcal{M}}_{g,n}$ are known to be extremal [CC15], as are sporadic strata in higher codimension. In [Bla21], I analyze morphisms between the moduli spaces of rational curves to establish new techniques to prove extremality in higher codimension, providing evidence for the following conjecture.

Conjecture 10 ([Bla21]). *All boundary strata in $\overline{\mathcal{M}}_{g,n}$ are extremal.*

As a consequence of the new techniques developed, I am able to show that a large family of *rational tails strata* – boundary strata whose dual graphs are trees – are extremal in their effective cones. I am also able to completely address the genus zero situation.

Theorem 11 ([Bla21]). *All boundary strata in $\overline{\mathcal{M}}_{0,n}$ are extremal.*

3. Future Work

3.1. **Alternative compactifications.** In addition to the ongoing work on additional compactifications of $\mathcal{M}_{g,n}$ and classification problems like Conjecture 8, I also have early computational evidence that the ψ-class intersection theory of more general modular compactifications of $\mathcal{M}_{g,n}$ can be expressed in a combinatorial way similar to that of Hassett spaces or Q-stable spaces. The partial classification given in [Smy13] is highly combinatorial in nature, and unifying the combinatorics of that classification with the combinatorics of ψ-class intersections is of great interest to me.

I am also interested in using Corollary 3 to derive recursions among κ-classes, a problem which has both theoretical and computational implications. Computationally, κ-class intersection numbers were available previously ([AC96]), but the algorithm's heavily recursive structure was problematic in high genus. Corollary 3 allows for a far more efficient computation. Being able to compute relations among κ-classes will also likely lead to extensions of results such as in [Pan12] for curves of compact type and help determine the structure of the κ-ring over all of $\overline{\mathcal{M}}_{g,n}$.

I have also begun to investigate the connections between Hassett spaces and parameter spaces of polygons in $\mathbb{R}^3$, an object of interest to mathematicians who work with random polygons. The space of equilateral polygons in $\mathbb{R}^3$ can be seen as the GIT quotient $(\mathbb{P}^1)^n /\!\!/ \mathrm{PGL}(2,\mathbb{C})$ ([Hu99]) which is birationally equivalent to $\overline{\mathcal{M}}_{0,n}$, or as a limit of Hassett spaces. There are several natural questions that arise from this connection, such as how the (well-studied) intersection theory of $\overline{\mathcal{M}}_{0,n}$ may be informative on polygon space, or if the Weil-Peterson metric on $\overline{\mathcal{M}}_{0,n}$ can be related to the natural metric on polygon space.

3.2. **Extremal classes.** The natural generalization of Theorem 9 to hyperelliptic classes in higher genus is still an open problem, but the techniques I use to handle the genus two case are largely inductive. Thus, given a finite number of base cases in each genus, the full result will follow. This requires a deeper understanding of hyperelliptic loci in general and remains an interesting area for future research. The situation for rational tails boundary strata is similar, but for arbitrary boundary strata, the theory becomes much more nuanced, and new techniques need to be developed. However, preliminary results show that the techniques which were successful in analyzing the hyperelliptic loci and boundary strata can be applied to some broader families of classes to better understand their cones as well.

Related to these topics is a broader question of what the effective cones of $\overline{\mathcal{M}}_{g,n}$ look like. Even in genus zero and codimension one, we are still lacking a complete picture: work in [CT15], [GK16], and [HKL18] has shown that $\overline{\mathcal{M}}_{0,n}$ is not a Mori dream space for $n \geq 10$; [CC14] and [Mul17b] show the same with $n \geq 3$ for $\overline{\mathcal{M}}_{1,n}$ and $\overline{\mathcal{M}}_{2,n}$ respectively. While a general answer still seems very far away, there are numerous cases that appear to be within reach.

References

[AC96] Enrico Arbarello and Maurizio Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli space of curves*, Journal of Algebraic Geometry **5** (1996), no. 4, 705–749. 5

[BB] Vance Blankers and Sebastian Bozlee, *A classification of Gorenstein compactifications of $\mathcal{M}_{1,n}$ with colliding markings*, in progress. 3, 4

[BC18] Vance Blankers and Renzo Cavalieri, *Intersections of ω classes in $\overline{\mathcal{M}}_{g,n}$*, Proceedings of Gökova Geometry–Topology Conference 2017 (2018), 37–52. 2

[BC20] Vance Blankers and Renzo Cavalieri, *Wall-Crossings for Hassett Descendant Potentials*, International Mathematics Research Notices (2020), rnaa077. 3

[BC21] Vance Blankers and Renzo Cavalieri, *Witten's conjecture and recursions for κ classes*, European Journal of Mathematics **7** (2021), 309–339. 3

[BKN21] Sebastian Bozlee, Bob Kuo, and Adrian Neff, *A classification of modular compactifications of the space of pointed elliptic curves by Gorenstein curves*, arXiv:2105.10582 (2021). 3, 4

[Bla20] Vance Blankers, *Hyperelliptic classes are rigid and extremal in genus two*, Épijournal de Géométrie Algébrique **4** (2020), no. 2. 4

[Bla21] ______, *Extremality of rational tails boundary strata in $\overline{\mathcal{M}}_{g,n}$*, arXiv:2002:12403 (2021). 4, 5

[CC14] Dawei Chen and Izzet Coskun, *Extremal effective divisors on $\overline{\mathcal{M}}_{1,n}$*, Mathematische Annalen **359** (2014), no. 3-4, 891–908. 4, 5

[CC15] ______, *Extremal higher codimension cycles on moduli spaces of curves*, Proceedings of the London Mathematical Society **111** (2015), no. 1, 181–204. 4

[Che18] Dawei Chen, *An extremal effective survey about extremal effective cycles in moduli spaces of curves*, Geometry of Moduli, Springer International Publishing, 2018, pp. 65–74. 4

[CMP$^+$20] Alois Cerbu, Steffen Marcus, Luke Peilen, Dhruv Ranganathan, and Andrew Salmon, *Topology of tropical moduli of weighted stable curves*, Advances in Geometry **20** (2020), no. 4, 445–462.

[CT15] Ana-Maria Castravet and Jenia Tevelev, *$\overline{M}_{0,n}$ is not a Mori dream space*, Duke Math. J. **164** (2015), no. 8, 3851–3878. 4, 5

[CT16] Dawei Chen and Nicola Tarasca, *Extremality of loci of hyperelliptic curves with marked Weierstrass points*, Algebra & Number Theory **10** (2016), no. 1, 1935–1948. 4

[DM69] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES **36** (1969), 75–109.

[EH87] David Eisenbud and Joe Harris, *The kodaira dimension of the moduli space of curves of genus $\geq$ 23*, Invent. Math. **90** (1987), no. 2, 359–387. 4

[GK16] José Luis González and Kalle Karu, *Some non-finitely generated Cox rings*, Compos. Math. **152** (2016), no. 5, 984–996. 5

[Has03] Brendan Hassett, *Moduli spaces of weighted pointed stable curves*, Advances in Mathematics **173** (2003), no. 2, 316–352. 2

[HKL18] Jürgen Hausen, Simon Keicher, and Antonio Laface, *On blowing up the weighted projective plane*, Mathematische Zeitschrift **290** (2018), 1339–1358. 5

[Hu99] Yi Hu, *Moduli spaces of stable polygons and symplectic structures on $\overline{\mathcal{M}}_{0,n}$*, Compositio Mathematica **118** (1999), no. 2, 159–187. 5

[Kon92] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix airy function*, Communications in Mathematical Physics **147** (1992), no. 1, 1–23. 3

[Log03] Adam Logan, *The Kodaira dimension of moduli spaces of curves with marked points*, Amer. J. Math. **125** (2003), no. 1, 105–138. 4

[Mul17a] Scott Mullane, *On the effective cone of higher codimension cycles in $\overline{\mathcal{M}}_{g,n}$*, Mathematische Zeitschrift (2017). 4

[Mul17b] ______, *On the effective cone of $\overline{\mathcal{M}}_{g,n}$*, Advances in Mathematics **320** (2017), 500–519. 4, 5

[Mul19] ______, *k-differentials on curves and rigid cycles in moduli space*, Preprint, arXiv:1905.03241 (2019). 4

[Mul20] ______, *Non-polyhedral effective cones from the moduli space of curves*, Preprint, arXiv:2001.08204 (2020). 4

[MZ00] Yuri I Manin and Peter Zograf, *Invertible cohomological field theories and weil-petersson volumes*, Annales de l'institut Fourier **50** (2000), no. 2, 519–535. 3

[Opi16] Morgan Opie, *Extremal divisors on moduli spaces of rational curves with marked points*, Michigan Math. J. **65** (2016), no. 2, 251–285. 4

[Pan12] Rahul Pandharipande, *The kappa ring of the moduli of curves of compact type*, Acta Math **208** (2012), 335–388. 5

[Smy13] David Ishii Smyth, *Towards a classification of modular compactifications of $\mathcal{M}_{g,n}$*, Inventiones mathematicae **192** (2013), 459–503. 3, 5

[Smy19] ———, *Intersections of ψ-classes on $\mathcal{M}_{1,n}(m)$*, Transactions of the American Mathematical Society **372** (2019), 8679–8707. 4

[Tar15] Nicola Tarasca, *Double total ramifications for curves of genus 2*, Int. Math. Res. Not. IMRN **19** (2015), 9569–9593. 4